CS Theory (Spring '25)

Test 2 Review Problems

Instructor: *Toniann Pitassi and William Pires*

1 Concepts to review

Be comfortable answering the following:

(a) What is P ?

(b) What is NP ?

(c) What is a decidable language ?

(d) What is a recognizable language ?

(e) What's the language of a TM ?

(f) What's the language of a NTM ?

2 True or False

(a) Every NP-hard language is NP-complete.

(b) Every decidable language is in NP.

(c) Every language in P is in NP.

(d) There exists a finite language that is undecidable.

(e) There exists an infinite language that is in P.

(f) There is an algorithm for $3SAT$ that runs in exponential-time.

(g) The following problem is decidable: on input $< M >$, accept if and only if M halts in polynomial-time on all inputs.

(h) A polynomial-time mapping reduction from $\mathsf{3SAT}$ to $L = \{0^n 1^n \mid n \geq 0\}$ implies $P = NP$.

(i) Let $\overline{HALT}$ be the set of inputs $< M, w >$ such that M does not halt on w. $\overline{HALT}$ is infinite.

(j) $\overline{HALT}$ is recognizable.

(k) Let A be polynomial-time mapping reducible to B. Then it is possible that A is in P but B is not in in NP.

(l) The set of languages over $\Sigma = \{0, 1\}$ is countable.

(m) The set of all strings $\langle M \rangle$ where M is a TM, is countable.

3 Recognizable languages

Prove the following languages are recognizable. In each problem you can assume the Turing machines take as inputs strings over $\Sigma = \{0,1\}$.

(a) $L = \{\langle M \rangle \mid\ M$ accepts at least one string w such that w has odd length and ends with a 1 $\}$

(b) $L = \{< M >\mid M$ accepts some string ending with 11 and one ending with $01\}$.

(c) $L = \{< M_1, M_2 >\mid$ such that there exists a string w accepted by M_1 and rejected by $M_2\}$

4 Decidable languages

For each of the following languages, state whether or not it is decidable and prove your answer. To prove a language is undecidable using a Turing reduction. Use A_{TM} or Halt or Empty for your reduction. To prove decidable give pseudocode to describe a decider. In each problem you can assume the Turing machines take as inputs strings over $\Sigma = \{0,1\}$.

(a) $L = \{\langle M \rangle \mid\ M$ accepts $00\}$.

(b) $L = \{\langle M \rangle \mid\ M$ accepts at least 3 strings $\}$

(c) $L = \{< M_1, M_2 >\mid$ there exists some string w such that M_1 accepts w and M_2 rejects w. $\}$

5 NP Completeness

Prove the following languages are NP-complete. For proving the language is in NP, you can either give a verifier or an NTM. For proving the language is NP-hard, you must use one of the following NP-complete languages in your reduction: 3SAT, Graph-Color, Clique, Independent Set, Hitting-Set, Subset-Sum. (Graph-Color is a generalization of 3Color. The input is a pair (G, k) where G is an undirected graph, and it is accepted iff there is a proper k-coloring of G.)

(a) You are given an k tables and a list of n guests. For each guest you must place at one of the tables (some tables can be left empty, each table can have up to n guests).

You're also given a list $C = \{c_1, \ldots, c_\ell\}$ of constraints of the form (g_i, g_j) which says that guests i and j cannot sit at the same table.

$\textsf{Floor-Plan} = \{\langle k, n, C\rangle \mid$ we can place the n guests using k tables and respect the constraints in $C.\}$

(b) Consider the $\textsf{Knapsack}$ Problem: You are given a set S of n items each with a value v_i and a weight w_i. You're also given a target t and a weight limit W. (all numbers are positive integers given in binary). Does there exists $T \subseteq S$ such that $\sum_{i\in T} v_i = t$ and $\sum_{i \in T} w_i \leq W$?

$$\textsf{Knapsack} = \{\langle S, t, W\rangle \mid \exists T \subseteq S \text{ such that } \sum_{i\in T} v_i = t \text{ and } \sum_{i\in T} w_i \leq W\}$$

(c) $\mathrm{Double-Clique} = \{\langle G, k\rangle \mid G$ is a graph that contains two disjoint cliques of size $k\}$ (the two cliques are not allowed to have any vertices in common).

(d) VertexCover $= \{\langle G, k\rangle \mid \exists$ a subset S of k vertices such that every edge has at least one endpoint in $S\}$. (I.e. for every edge (u, v) in the graph u or v (or both) is in S)

(e) 3SAT-under-$\rho = \{\langle \phi, j\rangle \mid \phi$ is a 3-CNF formula and it has a satisfying assignment α which sets $x_j = 1\}$. Here ϕ has n variables $x_1, \ldots, x_n$ and you can assume $\langle \phi, j\rangle$ always has $1 \leq j \leq n$.

(f) FIND2 $-$ 3SAT $= \{\langle \phi\rangle \mid \phi$ is a 3-CNF formula and it has at least 2 satisfying assignments$\}$

(g) William needs to solve the following ExamDesign problem. He has a list of problems, and he knows for each problem which students will really enjoy that problem. He needs to choose a subset of problems for the exam such that for each student in the class, the exam includes at least one question that student will really enjoy. On the other hand, he does not want to spend the entire summer grading an exam with dozens of questions, so the exam must also contain as few questions as possible. Prove that the ExamDesign problem is NP-hard.